Mon. Not. R. Astron. Soc. **000**, 000–000 (0000) Printed 1 February 2008 (MN LATEX style file v1.4)

Non-adiabatic tidal forcing of a massive, uniformly rotating star III: Asymptotic treatment for low frequencies in the inertial regime

J.C. Papaloizou [1] and G.J. Savonije [2]

[1] *Astronomy Unit, School of Mathematical Sciences, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, UK*

[2] *Astronomical Institute 'Anton Pannekoek', University of Amsterdam and Centre for High Energy Astrophysics (CHEAF), Kruislaan 403, 1098 SJ Amsterdam, The Netherlands*

1 February 2008

ABSTRACT

We describe a generalization of the asymptotic calculation of the tidal torques experienced by a massive star as a result of a companion in circular orbit originally considered by Zahn (1975,1977) to the case of a rotating star when the forcing frequency is small and in the inertial regime, that is it is less than twice the rotation frequency in magnitude. The results confirm the presence of a strong toroidal mode resonance feature for retrograde forcing and also, with a simple description of the convective core, the presence of some core inertial mode features in the response. These were found numerically by Savonije and Papaloizou (astro-ph/9706186).

Key words: Hydrodynamics– Stars: binaries– Stars: rotation– Stars: oscillation – Stars: tides

1 INTRODUCTION

In two recent papers, (Savonije, Papaloizou and Alberts, 1995: SPA, Savonije and Papaloizou, 1997: SP) we studied the response of a uniformly rotating massive star to the tidal forcing due to a companion in a circular orbit (e.g. a massive X-ray binary). To make the problem tractable, centrifugal forces were neglected, allowing a spherically symmetric equilibrium, but Coriolis forces were retained allowing normal modes governed by rotation to enter into the response. This problem is complex because the dense spectrum of normal modes causes excitation of short wavelength responses which lead to numerical difficulties. As a result the analysis of SPA could not be extended into the inertial regime in which the forcing frequency is less than twice the rotation frequency in magnitude. In SP numerical results appropriate to the inertial regime for modest stellar rotation rates have been obtained through the introduction of a viscosity which provides numerical damping of the shortest wavelengths in the response in convective regions.

The first calculations of the tidal torques experienced by a non-rotating massive star as a result of a companion in a circular orbit were performed by Zahn (1975,1977) who developed an asymptotic approach valid for low forcing frequencies. Papaloizou and Savonije (1984, 1985) found that this could give reasonable results provided the star was unevolved with the convective core not being too small in size.

In this paper we investigate a generalization of the asymptotic calculation of the tidal torques to the case of a rotating star when the forcing frequency is small in magnitude and in the inertial regime. Such an analysis is of interest in order to complement the numerical work which runs into difficulties because of the short wavelength nature of the response at low frequencies. We find that some of the phenomena found by SP, in particular the strong toroidal mode resonance, is present in the asymptotic analysis as well and persists in the low frequency limit. This may be important for retrograde forcing leading to a more rapid synchronization of spin and orbit than would be expected from consideration of non-rotating stars.

In section 2 we give the basic equations for a uniformly rotating star subject to tidal perturbation due to a companion in circular orbit, formulating a single equation giving the response of the adiabatic interior. In section 3 we consider the response of the convective core, indicating how inertial modes may be excited there. We also consider the radiative exterior indicating how use of the traditional approximation leads to a separable problem for excited generalized g and r modes in that region. In

section 4 we consider the WKB approximation for the radiative region with outgoing wave conditions and how toroidal mode resonances lead to a global response. We go on to evaluate the action of the tidal torque in producing an angular momentum flux through wave excitation in section 5. In section 6 we describe numerical results while in section 7 we summarise the conclusions.

2 BASIC EQUATIIONS

We consider a uniformly rotating massive secondary star with mass M_s and radius R_s. We assume the angular velocity of rotation $\vec{\Omega_s}$ to be much smaller than the break up speed, i.e. $(\Omega_s/\Omega_c)^2 \ll 1$, with $\Omega_c^2 = GM_s/R_s^3$, so that effects of centrifugal distortion ($\propto \Omega_s^2$) may be neglected in first approximation.
We use spherical coordinates (r, θ, φ), with origin at the secondary's centre, whereby $\theta = 0$ corresponds to its rotation axis which we assume to be parallel to the orbital angular momentum vector. However, we take the coordinates to be rotating with the secondary.
The linearized hydrodynamic equations governing the non-adiabatic response of the uniformly rotating star to the perturbing potential Φ_T are the equation of motion

$$\frac{\partial \mathbf{v}'}{\partial t} + 2\Omega_s \mathbf{k} \times \mathbf{v}' = -\frac{1}{\rho}\nabla P' + \frac{\rho'}{\rho^2}\nabla P - \nabla\Phi_T, \tag{1}$$

and the equation of continuity

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho \mathbf{v}') = 0 \tag{2}$$

Here the velocity perturbation is $\mathbf{v}'$ and the pressure and density perturbations are P' and ρ' respectively. The unit vector along the rotation axis is denoted by $\mathbf{k}$.
We consider a close binary system in which the orbit is circular with angular velocity ω and orbital separation D. The dominant tidal term of the primary's perturbing potential is then given by the real part of:

$$\Phi_T = fr^2\, P_2^2(\mu)\ \exp(2i[\omega_f t - \varphi]) \equiv \Phi_{T0}\ \exp(2i[\omega_f t - \varphi]) \tag{3}$$

where M_p is the companion's mass, $\omega_f = \omega - \Omega_s$ is the relative orbital frequency, as seen in the rotating frame, $\mu = cos\theta$, $P_2^2(\mu)$ is the associated Legendre polynomial for $l = |m| = 2$ and

$$f = -\frac{GM_p}{4D^3}.$$

For simplicity we have adopted the Cowling approximation, i.e. we have neglected perturbations to the secondary's gravitational potential caused by tidal distortion. This approximation is reasonable because of the high central condensation.
In this paper we adopt a simple model for calculating the tidal response in which the angular momentum exchange between the rotating star and the binary orbit occurs through angular momentum carying waves that are excited in the region of the convective core boundary and the radiative layers in its neighbourhood. It is supposed that these waves are subsequently dissipated with the consequent angular momentum exchange (see Papaloizou and Savonije 1984,1985, and Goldreich and Nicholson 1989). In order to analyse this model, only the interior regions of the star need to be considered. Then the perturbations are (almost) adiabatic such that

$$\frac{\partial P'}{\partial t} - \frac{\Gamma P}{\rho}\frac{\partial \rho'}{\partial t} + \mathbf{v}' \cdot \left(\nabla P - \frac{\Gamma P}{\rho}\nabla\rho\right) = 0. \tag{4}$$

For simplicity in what follows below we shall take Γ to be constant and equal to 5/3. Then (2) and (4) can be combined to give

$$\frac{\partial P'}{\partial t} = -\Gamma P^{(1-1/\Gamma)}\nabla \cdot (P^{1/\Gamma}\mathbf{v}'). \tag{5}$$

2.1 Reduction of the Response Equations

The linear response to the tidal potential (3) is such that all perturbations have a φ and t dependence through a factor $\exp(2i[\omega_f t - \varphi])$. But note that, due to the occurrence of the Coriolis terms in the equations, the solution is no longer separable in θ, as it is for non-rotating stars. Equations (1) and (2) give for the components of the displacement $\boldsymbol{\xi} = (i\sigma)^{-1}\mathbf{v}'$:

$$-\sigma^2\xi_r - 2i\Omega_s\sigma\sin\theta\xi_\varphi = -K\frac{\partial W}{\partial r} - (\xi_r + \Phi_T/g)\mathcal{A}, \tag{6}$$

$$-\sigma^2\xi_\theta - 2i\Omega_s\sigma\cos\theta\xi_\varphi = -\frac{K}{r}\frac{\partial W}{\partial\theta}, \tag{7}$$

$$-\sigma^2\xi_\varphi + 2i\Omega_s\sigma\left(\cos\theta\xi_\theta + \sin\theta\xi_r\right) = -\frac{K}{r\sin\theta}\frac{\partial W}{\partial\varphi}. \tag{8}$$

Here $W = (P' + \rho\Phi_T)/P^{1/\Gamma}$, and $K = P^{1/\Gamma}/\rho$. The square of the Brunt-Vaisala frequency is given by

$$\mathcal{A} = \frac{\nabla P}{\rho}\cdot\left(\frac{\nabla\rho}{\rho} - \frac{\nabla P}{\Gamma P}\right),$$

the local acceleration due to gravity is given by $g = -\rho^{-1}(dP/dr)$, and the forcing frequency $\sigma = 2\omega_f$. We can use equations (6) - (8) to express $\boldsymbol{\xi}$ in terms of W. Using the notation $(\xi_r, \xi_\theta, \xi_\varphi) \equiv (\xi_1, \xi_2, \xi_3)$, $(h_1, h_2, h_3) \equiv (1, 1/r, 1/(r\sin\theta))$,and $(x_1, x_2, x_3) \equiv (r, \theta, \varphi)$, we may write (using the summation convention)

$$\xi_i = \frac{A_{ij}}{\Delta}\left(Kh_j\frac{\partial W}{\partial x_j} + \delta_{j1}\frac{\Phi_T}{g}\mathcal{A}\right). \tag{9}$$

The components of the Hermitian (for real σ) matrix $[A]$ are given by......

$$A_{11} = -\sigma^4 + 4\sigma^2\Omega_s^2\cos^2\theta,\quad A_{12} = -4\Omega_s^2\sigma^2\sin\theta\cos\theta,\quad A_{13} = 2i\Omega_s\sigma^3\sin\theta$$

$$A_{22} = \sigma^2(-\sigma^2 + \mathcal{A}) + 4\sigma^2\Omega_s^2\sin^2\theta,\quad A_{23} = -2i\Omega_s\sigma\cos\theta(-\sigma^2 + \mathcal{A}),\quad A_{33} = \sigma^2(-\sigma^2 + \mathcal{A}),$$

with the unspecified components being given by the Hermitian condition assuming σ to be real and

$$\Delta = (-\sigma^2 + \mathcal{A})(\sigma^4 - 4\sigma^2\Omega_s^2\cos^2\theta) + 4\sigma^4\Omega_s^2\sin^2\theta.$$

Using the above expression for the components of the displacement in (5) gives a single second order partial differential equation for W in the form

$$P^{1/\Gamma}W - \rho\Phi_T = -\Gamma P^{(1-1/\Gamma)}\frac{h_i}{q_i}\frac{\partial}{\partial x_i}\left(P^{1/\Gamma}q_i\frac{A_{ij}}{\Delta}\left(Kh_j\frac{\partial W}{\partial x_j} + \delta_{j1}\frac{\Phi_T}{g}\mathcal{A}\right)\right), \tag{10}$$

where $(q_1, q_2, q_3) = (r^2, \sin\theta, 1)$.
As the φ and t dependence of W is through a separable factor $\exp(im(\omega_f t - \varphi))$, with $m \equiv 2$, after the replacement $\frac{\partial}{\partial\varphi} \to -im$ and $\Phi_T \to \Phi_{T0}$, equation (10) becomes a second order partial differential equation of mixed type for W as a function of r and θ. It is hyperbolic whenever a real wave vector $(k_1, k_2, 0)$ exists such that $A_{ij}k_ik_j = 0$. This condition is equivalent to the requirement that σ satisfies the local dispersion relation (see Tassoul 1978, SPA)

$$\sigma^2 = \frac{k_2^2\mathcal{A} + 4\Omega_s^2(k_1\cos\theta - k_2\sin\theta)^2}{(k_1^2 + k_2^2)}. \tag{11}$$

When this can be satisfied and the boundary conditions are non-dissipative, one expects a dense spectrum of normal modes and a singular response to forcing (SPA). This problem is avoided here by, from now on, allowing σ to have a small negative imaginary part. This latter Landau prescription corresponds to the forcing potential being slowly switched on at time $t = -\infty$. This also leads naturally to the selection of a predominantly outgoing wave boundary condition, corresponding to the physical situation where the surface regions of the star are assumed to be highly dissipative with little or no wave reflection occurring from them.

3 ASYMPTOTIC TREATMENT FOR LOW FREQUENCIES

Because of lack of separability, the solution of equation (10) cannot be undertaken analytically in general. However, progress can be made if we assume that the square of the forcing frequency σ^2 is small compared to the $G\rho$, the square of the inverse of the local dynamical timescale. We shall be specially interested in the inertial regime and so we shall assume σ^2/Ω_s^2 is comparable to unity.

3.1 The radiative zone

In a radiative region $\mathcal{A}$ is non zero and positive and for low frequencies $\sigma^2/\mathcal{A}$ can be considered to be small, apart possibly from a region in the vicinity of the interface between the convective core and surrounding radiative zone. However, what follows below can be shown to be valid also in such a region. We write equation (10) in the form

$$P^{1/\Gamma}W + \Gamma P^{(1-1/\Gamma)}\frac{h_i}{q_i}\frac{\partial}{\partial x_i}\left(P^{1/\Gamma}q_i\frac{A_{ij}}{\Delta}Kh_j\frac{\partial W}{\partial x_j}\right) = S, \tag{12}$$

where we have grouped the forcing terms involving Φ_{T0} together on the right hand side of (12) in S. Considering that $\epsilon \equiv \sigma^2/\mathcal{A}$ defines a small parameter, we may attempt to find solutions by expanding the left hand side of (12) to first order in this parameter. We also allow the solution to vary rapidly with radius such that $\frac{\partial}{\partial r} = O(\epsilon^{-1/2})\frac{1}{r}$. It is further assumed that the

angular variation is much less than the radial variation which is the situation for the expected excited low frequency g modes. Then expansion of (12) in powers of ϵ gives to zero order

$$P^{1/\Gamma}W\sigma^2 - \frac{\sigma^2\Gamma P^{(1-1/\Gamma)}}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2KP^{1/\Gamma}}{\mathcal{A}}\frac{\partial W}{\partial r}\right) + \frac{K\Gamma P}{r^2}O_\perp(W) = \sigma^2 S, \tag{13}$$

where the operator $O_\perp$ is defined through

$$O_\perp(W) = -\frac{\partial Q}{\partial \mu} + \frac{2m\Omega_s\mu Q}{\sigma(1-\mu^2)} - \frac{m^2 W}{(1-\mu^2)}, \tag{14}$$

with Q being related to W through

$$Q\left(1 - \frac{4\Omega_s^2\mu^2}{\sigma^2}\right) = -(1-\mu^2)\frac{\partial W}{\partial \mu} - \frac{2m\Omega_s\mu W}{\sigma}. \tag{15}$$

3.2 The convective core

We idealize the convective core to be a region in which $\mathcal{A} = 0$ interior to the boundary, where we suppose $\mathcal{A} = 0$, but $\nabla\mathcal{A} \neq 0$. When $\mathcal{A} = 0$, we cannot make an expansion based on the smallness of $\sigma^2/\mathcal{A}$. In this case we must retain (12). However, S takes the simple form $S = \rho\Phi_{T0}$, and thus (12) becomes

$$P^{1/\Gamma}W + \Gamma P^{(1-1/\Gamma)}\frac{h_i}{q_i}\frac{\partial}{\partial x_i}\left(P^{1/\Gamma}q_i\frac{A_{ij}}{\Delta}Kh_j\frac{\partial W}{\partial x_j}\right) = \rho\Phi_{T0}. \tag{16}$$

Noting that $\mathcal{A} = 0$, the unforced form of (16) gives rotationally governed inertial modes in the core (see Papaloizou and Pringle 1981, SPA). The local dispersion relation is (11) with $\mathcal{A} = 0$. In order to solve (16) boundary conditions for W are needed. In the low frequency limit we may use the fact that the stratification in the radiative zone leads to ξ_r being small compared to the other components of the displacement. This is a familiar property of oscillation modes such as g modes in radiative layers. This suggests the boundary condition $\xi_r = 0$ at the convective core boundary. Using (9), this can be expressed as a condition on W.

For real σ finding W in the convective core is problematic because of the dense spectrum of the inertial modes (Greenspan 1968, Papaloizou and Pringle 1981, SPA). However, if σ is complex as we shall assume, singular responses are avoided except possibly when there are near resonances with global modes. A similar situation is expected if viscosity is introduced. Solving (16) subject to $\xi_r = 0$ on the convective core boundary is equivalent to solving an elliptic boundary value problem and the solution will provide W on the core boundary which may be used as a boundary condition there for the problem of finding W in the radiative zone.

The elliptic boundary value problem described above is not simply soluble in general, even in the low frequency limit. However, an exception occurs when the core is assumed to have constant density and pressure, an assumption which is only justifiable when the core is small. Then a solution can be found in the form of a finite polynomial. To find such solutions, it is best to write equation (16) in cylindrical polar coordinates (ϖ, φ, z) in the form

$$\frac{\rho\sigma^2}{\Gamma P}W + \frac{1}{(1-x^2)}\left[\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial W}{\partial\varpi}\right) - \frac{m^2W}{\varpi^2}\right] + \frac{\partial^2 W}{\partial z^2} = \frac{3f\rho^2\sigma^2}{\Gamma P^{(1+1/\Gamma)}}\varpi^2, \tag{17}$$

where

$$x = \frac{2\Omega_s}{\sigma}.$$

In order to proceed we take P and ρ to be constant and equal to their values on the convective core boundary. In the low frequency limit the first term on the left hand side of (17) may be neglected. The problem is then equivalent to calculating the response of an homogeneous incompressible sphere (see Greenspan, 1968). In this case the solution that has zero normal displacement at the core boundary can be written as a polynomial in ϖ and z for $m = 2$ in the form

$$W = a_c\varpi^4 + b_c\varpi^2 + c_c z^2\varpi^2, \tag{18}$$

where

$$a_c = \frac{3f\rho^2\sigma^2}{2\Gamma P^{(1+1/\Gamma)}}\frac{(1-x^2)(2+x)}{(14-x^2+7x)},$$

$$b_c = -\frac{3f\rho^2 r_c^2\sigma^2}{2\Gamma P^{(1+1/\Gamma)}}\frac{(1+x)(4-x^2)}{(14-x^2+7x)}, \quad \text{and}$$

$$c_c = \frac{3f\rho^2\sigma^2}{2\Gamma P^{(1+1/\Gamma)}}\frac{(1+x)(2-x)}{(14-x^2+7x)},$$

where r_c is the radius of the convective core. We remark that the response is resonant when $14 - x^2 + 7x = 0$. There are thus only two real resonant forcing frequencies corresponding to the roots of the quadratic given by $x = (7 \pm \sqrt{(105)})/2$, or $\omega_f/\Omega_s = (\pm\sqrt{(105)} - 7)/28$. Although the spectrum is dense, the nature of the forcing potential allows only two potential resonances to be excited in this case. It would seem reasonable to assume that other modes can only be weakly excited in more general cases. In fact the above solution for a uniform core may be regarded as the first member of a sequence of higher order polynomial approximations to solutions in more general cases (see Papaloizou and Pringle, 1981). Accordingly we shall use (18) to determine W on the core surface in the analysis presented below.

4 ASYMPTOTIC SOLUTION IN THE RADIATIVE ZONE

The governing equation (13) developed for small $\sigma^2/\mathcal{A}$ can be solved by utilising the fact that the operator on the left hand side is separable (see SPA). We may write

$$W = \sum_{n=0}^{\infty} w_n(r) X_n(\mu). \tag{19}$$

Here the X_n are a set of orthogonal functions of μ over $(-1, 1)$ so that the expansion coefficients are given by

$$w_n = \frac{\int_{-1}^{1} W X_n d\mu}{\int_{-1}^{1} X_n^2 d\mu}.$$

The X_n satisfy a second order ordinary differential equation defined through the first order pair (see SPA)

$$-\frac{\partial \mathcal{D}_n}{\partial \mu} + \frac{mx\mu\mathcal{D}_n}{(1-\mu^2)} - \frac{m^2 X_n}{(1-\mu^2)} = -\lambda_n X_n, \tag{20}$$

$$\mathcal{D}_n\left(1 - x^2\mu^2\right) = -(1-\mu^2)\frac{\partial X_n}{\partial \mu} - mx\mu X_n. \tag{21}$$

The eigenvalues λ_n are determined so as to make X_n regular at $\mu = \pm 1$. Note that they are functions of the parameter x and thus the forcing frequency and the stellar rotation rate. Note too that for potentials of the type we consider here which are even functions of μ, we may restrict ourselves to λ_n such that $X_n, \mathcal{D}_n$ are even and odd functions of μ respectively.
Using (19) we obtain

$$O_\perp(W) = -\sum_{n=0}^{\infty} \lambda_n w_n(r) X_n(\mu). \tag{22}$$

Using the expansion given by equation (19) and the orthogonality of the X_n, equation (13) gives

$$w_n\left(\lambda_n - \frac{\sigma^2\rho r^2}{\Gamma P}\right) + \frac{\sigma^2}{KP^{1/\Gamma}}\frac{\partial}{\partial r}\left(\frac{r^2 K P^{1/\Gamma}}{\mathcal{A}}\frac{\partial w_n}{\partial r}\right) = -\frac{\sigma^2\rho r^2 S_n}{\Gamma P^{1+1/\Gamma}}, \tag{23}$$

where

$$S_n = \frac{\int_{-1}^{1} S X_n d\mu}{\int_{-1}^{1} X_n^2 d\mu}.$$

The problem is thus reduced to solving a set of second order ordinary differential equations for $w_n(r)$. This is possible because of the fact that the normal mode problem becomes separable in the traditional approximation (see Chapman and Lindzen, 1970 for a discusion in the context of atmospheric tides and also SPA). This approximation neglects the θ component of the stellar angular velocity (Unno et al 1989), and it is expected to become valid in the stratified radiative layers in the limit of low ferquencies when the radial displacement becomes small compared to the angular displacements.

4.1 Reduction of S

From direct calculation using equation (10) we obtain for large $\mathcal{A}$

$$S = \frac{\Gamma P^{1-1/\Gamma}}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2 P^{1/\Gamma}\Phi_{T0}}{g}\right) - \frac{x^2\Gamma P}{gr}\frac{\partial}{\partial \mu}\left(\frac{\mu(1-\mu^2)\Phi_{T0}}{(1-x^2\mu^2)}\right) + \frac{mx\Gamma P\Phi_{T0}}{gr(1-x^2\mu^2)} + \rho\Phi_{T0}, \tag{24}$$

and after performing an integration by parts

$$S_n \int_{-1}^{1} X_n^2 d\mu = \int_{-1}^{1} S X_n d\mu = f\rho r^2\left(\mathcal{I}\frac{\Gamma P^{1-1/\Gamma}}{\rho r^4}\frac{d}{dr}\left(\frac{r^4 P^{1/\Gamma}}{g}\right) - \frac{x^2\Gamma P}{\rho g r}\mathcal{J} + \mathcal{I}\right), \tag{25}$$

with

$$\mathcal{I} = \int_{-1}^{1} X_n P_2^2(\mu) d\mu,$$

and

$$\mathcal{J} = \int_{-1}^{1} (\mu \mathcal{D}_n - m X_n/x) P_2^2(\mu) d\mu.$$

4.2 Behaviour of the eigenvalues $\lambda_n(x)$

The calculation of the tidal torque has thus been reduced to solving the sets of second order differential equations (23). The boundary conditions are that the solution should, if possible, correspond to a predominantly outgoing wave at large radii, and w_n should be matched at the core boundary to the solution obtained above that applies to the convective core.

Clearly the form of the solutions of the determining equation (23) depends on the eigenvalues λ_n. ¿From general scaling arguments one expects $w_n = O(\sigma^2/\lambda_n)$ which vanishes as σ vanishes. When $\lambda_n \gg 0$, the unforced solutions for w_n are short wavelength g mode like waves. There is special interest in the smallest positive λ_n because these lead to solutions with the longest radial wavelengths which will give the strongest responses to global forcing of the type considered here.

For a non-rotating star ($x = 0$) the eigenvalues $\lambda_n = l(l+1)$, with $l \equiv n = 2, 4, 6...$ and the corresponding eigenfunctions $X_n = P_l^2(\mu)$. However, the behaviour of λ_n is very different in the inertial regime for which $-1 < x^{-1} < 1$. We plot the smallest positive λ_n we obtained for a range of values of x in the inertial regime in figure 1. It will be seen that there is a significant difference between positive and negative x. For $x > 0$, corresponding to prograde relative rotation of the companion, λ_n is similar to the non-rotating case. However, for $x < 0$, corresponding to retrograde relative rotation, λ_n becomes large for $x^{-1} < -1/6$. These eigenvalues correspond to disturbances confined cosely to the equator. For $-1/6 < x^{-1} < 0$, another small λ_n exists corresponding to a toroidal mode resonance. These eigenvalues are important because they lead to solutions with strong global responses to the forcing tide.

4.3 Toroidal mode resonances

Strict toroidal mode resonance can be defined to occur when x is such that an eigenvalue $\lambda_n = 0$, (see SPA). Then we have $x = -(l(l+1))/m$, with $l \equiv n \geq |m|$ being an odd integer to provide solutions with the required symmetry type. These resonances corespond to the r mode frequencies (Papaloizou and Pringle, 1978)

$$\sigma = \frac{-2m\Omega_s}{l(l+1)}.$$

The corresponding eigenfunctions are $\mathcal{D}_n = P_l^{|m|}(\mu)$, and $X_n = m^{-2}(-l(l+1)\mu P_l^{|m|}(\mu) - (1-\mu^2) dP_l^{|m|}(\mu)/d\mu)$, where $P_l^{|m|}(\mu)$ denotes the standard Legendre function. The lowest order resonance has $l = 3$ so that $x^{-1} = -1/6$.

We see that, for positive values of m , the toroidal mode resonances occur for negative forcing frequencies, corresponding to retrograde orbital rotation relative to the star. In this case the star rotates faster than the orbit so that the action of the tides is to cause the star to spin down. It is clear from the above discussion that the existence of toroidal r mode resonances leads to a qualitatively different tidal response for some negative forcing frequencies which is in general much larger than for corresponding positive forcing frequencies. The tidal evolution timescale will accordingly be shorter in that case.

5 WKB SOLUTION

It is possible to solve equation (23) by a Green's function method as in Papaloizou and Savonije (1985). In general extended regions outside the convective core are found to contribute to the excitation of an outgoing wave. However, for simplicity we here look at the solution of equation (23) in the low frequency limit where a WKB approximation should be adequate. In this case the largest lenghthscale associated with the solution for w_n occurs near the convective core boundary (Zahn, 1977) where, for $\lambda_n > 0$, an outgoing wave is excited. Only this region matters for wave excitation in the limit of low forcing frequencies but if this limit is to be of practical use, the convective core should not be too small.

To construct the solution, we write $z = r - r_c$, with r_c being the radius of the convective core boundary. Further we assume a local first order Taylor expansion such that $\mathcal{A} = (\mathcal{A}')_c z$, and evaluate all other quantities in (23) at the core boundary. We then obtain

$$w_n \left(\lambda_n - \left(\frac{\sigma^2 \rho r^2}{\Gamma P} \right)_c \right) + \left(\frac{\sigma^2 r^2}{\mathcal{A}'} \right)_c \frac{\partial}{\partial z} \left(\frac{1}{z} \frac{\partial w_n}{\partial z} \right) = - \left(\frac{\sigma^2 \rho r^2 S_n}{\Gamma P^{1+1/\Gamma}} \right)_c . \tag{26}$$

Here the subscript c denotes evaluation at the core boundary. From now on we shall take this as read for all state variables and the subscript c will be dropped.
It is straightforward to express the solution of (26) that coresponds to only outward going waves in terms of Hankel functions, $H_\nu^{(1)}$, as follows. If $Q = z^{-1}dw_n/dz$, then the solution for Q is given by

$$Q = \frac{3^{5/6}\Gamma(4/3)\exp(-i\pi/6)}{2a^{1/6}} C_0 z^{1/2} H_{1/3}^{(1)}\left(\frac{2}{3}a^{1/2}z^{3/2}\right), \tag{27}$$

where

$$a = \frac{\mathcal{A}'}{\sigma^2 r^2}\left(\lambda_n - \frac{\sigma^2\rho r^2}{\Gamma P}\right), \tag{28}$$

$$C_0 = -aw_n(0) - \frac{S_n\rho\mathcal{A}'}{\Gamma P^{1+1/\Gamma}}, \tag{29}$$

with $w_n(0)$ denoting the prescribed value of w_n on the core boundary.
Here it is assumed that the real part of the forcing frequency is positive. Solutions appropriate to negative forcing frequencies can be found by setting $m \equiv 2 \rightarrow -m$, below while retaining the forcing frequency as positive.
The $w_n(0)$ are given by

$$w_n(0) = \frac{\int_{-1}^{1} W X_n d\mu}{\int_{-1}^{1} X_n^2 d\mu},$$

with W being evaluated on the core boundary. Using the solution given by (18), we obtain

$$w_n(0) = \frac{\int_{-1}^{1}(a_c r^4(1-\mu^2)^2 + b_c r^2(1-\mu^2) + c_c r^4\mu^2(1-\mu^2))X_n d\mu}{\int_{-1}^{1} X_n^2 d\mu},$$

5.1 Angular momentum flux and tidal torque

The asymptotic outgoing wave solutions outlined above are associated with a conserved angular momentum flow or wave action (Goldreich an Nicholson, 1989). Assuming the outgoing waves are ultimately absorbed near the surface, this angular momentum is deposited there and correspondingly removed from the orbit. But note that this exchange can be of negative sign for retrograde forcing. Thus in this picture, the tidal torque acts through the production and absorbtion of angular momentum carrying waves.
The radial component of the wave angular momentum flux appropriate for responses to the complex forcing potential is given by

$$F = \frac{m}{2}\mathcal{IM}\left(P'\xi_r^*\right),$$

where $\mathcal{IM}$ denotes the imaginary part (see Ryu and Goodman, 1992, Lin et al, 1993). Using

$$\xi_r = -\frac{K}{\mathcal{A}'}QX_n, \text{ and } P' = w_n X_n P^{1/\Gamma}$$

in the asymptotic limit, we find the asymptotic form (for large z) of the angular momentum flux associated with each λ_n

$$F = \frac{3^{8/3}\left(\Gamma(4/3)\right)^2 KmP^{1/\Gamma}\sigma^2 r^2|C_0|^2 X_n^2}{8\pi a^{1/3}(\mathcal{A}')^2\left(\lambda_n - \left(\frac{\sigma^2\rho r^2}{\Gamma P}\right)\right)}. \tag{30}$$

The total rate of change of angular momentum of the star is found by integrating the radial flux over a spherical surface at the convective core boundary. It is thus given by

$$\dot{J} = 2\pi r^2\int_{-1}^{1} F d\mu.$$

Using this, while neglecting $(\sigma^2\rho r^2)/(\Gamma P)$ in comparison to λ_n, we find after some algebraic manipulation that for a particular λ_n,

$$\dot{J} = \rho r^5\epsilon_T\Omega_c^2\frac{3^{8/3}\left(\Gamma(4/3)\right)^2\Theta^2|\Xi|^2}{2^{7/3}\lambda_n^{4/3}I_n}\left(\frac{|\omega_f|}{\Omega_c}\right)^{8/3}\left(\frac{\Omega_c^2\rho r^2}{\Gamma P}\right)^2\left(\frac{\Omega_c^2}{r\mathcal{A}'}\right)^{1/3}, \tag{31}$$

Here

$$\epsilon_T = \frac{M_p^2 R_s^6}{M_s^2 D^6},$$

and Θ is defined through

$$S_n = -\frac{f\rho r^2}{I_n}\left[\mathcal{I}\left(\frac{\Gamma P}{\rho g r}\left(\frac{4\pi\rho r^3}{M} - 6\right)\right) + \mathcal{J}\frac{x^2\Gamma P}{\rho g r}\right] \equiv \frac{f\rho r^2}{I_n}\Theta,$$

with M being the mass interior to radius r and

$$I_n = \int_{-1}^{1} X_n^2 d\mu.$$

The quantity Ξ is defined by

$$\Xi = 1 + \frac{w_n(0)\lambda_n \Gamma P^{1+1\Gamma} I_n}{f\rho^2 r^4 \sigma^2 \Theta},$$

and $\Omega_c^2 = GM_s R_s^{-3}$.
Here we have set $m = 2$, corresponding to prograde forcing frequencies. In this case $\dot{J} > 0$, corresponding to stellar spin up. For negative forcing frequencies the same expression may be used but the sign of $\dot{J}$ is reversed. In general we should sum over λ_n. However, for simplicity we shall restrict ourselves to the values displayed in figure 1 for the numerical work described below.
In order to compare with the work of SP, we introduce t_0 defined through

$$t_0 = \frac{kM_s R_s^2 \epsilon_T |\omega_f|}{\dot{J}},$$

with $kM_s R_s^2$ being the stellar moment of inertia. Thus

$$t_0 = \frac{kM_s R_s^2}{\Omega_c \rho r^5} \frac{2^{7/3}\lambda_n^{4/3} I_n}{3^{8/3}\left(\Gamma(4/3)\right)^2 \Theta^2 |\Xi|^2} \left(\frac{|\omega_f|}{\Omega_c}\right)^{-5/3} \left(\frac{\Omega_c^2 \rho r^2}{\Gamma P}\right)^{-2} \left(\frac{\Omega_c^2}{r\mathcal{A}'}\right)^{-1/3}. \tag{32}$$

We remark that for a particular stellar model $\Theta^2|\Xi|^2 = F(x)$ depends on the forcing and stellar rotation frequencies through the quantity x only. Thus we have the scaling law that

$$t_0 \propto (|\omega_f|)^{-5/3}/F(x), \tag{33}$$

which, as long as our simple prescription for including damping at inertial core mode resonances is used (see below), enables results to be scaled to different forcing frequencies for the same value of x.

6 NUMERICAL EVALUATION

We have evaluated t_0 using (32) for the 20 $M_\odot$ stellar model considered in SP. We display the result for $\log_{10} t_0$ plotted against ω_f/Ω_s for $\Omega_s = 0.1\Omega_c$ in the inertial regime in figure 2. But note that these results may be scaled to other values of Ω_s using the scaling relation (33).
Although we do not reproduce all the details found in SP, the most important features where they can be compared, including the form of the difference between prograde and retrograde forcing, and agreement with the values of t_0, to order of magnitude, are found. This type of qualitative agreement is reasonable in view of the large amount of variation in t_0 and the fact that the asymptotic analysis is carried out in the limit of both low forcing and rotation frequencies. This results in extreme sensitivity to the location of the convective core boundary. Also simplifying approximations were made to find the response of the convective core. Note too that the assumption of an outward going wave condition in the low frequency limit has prevented the appearance of the oscillatory behaviour found in SP. It is hoped that a future extension of the analysis will enable relaxation of these approximations.

We do find a strong toroidal mode resonance at the expected location $\omega_f = -\Omega_s/6$. This occurs because of the long wavelength response near this location resulting from the small λ_n. We remark that the assumption of an outgoing wave condition will break down in the centre of this resonance where the response is of very long wavelength. Also we find rather small values of t_0 near $x^{-1} = 0$ (note the resolution here was 0.01) in a domain that could not be considered in SP. The origin of this behaviour can be traced to the forcing term $\propto x^2$ in (25).
As in SP, we obtain larger t_0 in general for $\omega_f < -\Omega_s/6$ because of the larger λ_n. Note too that we have two core inertial mode resonance features at $\omega_f \sim -0.6\Omega_s$ and $\omega_f \sim 0.12\Omega_s$. Similar features were seen in SP. To obtain features of similar scale, we incorporated damping in the core by adding a negative imaginary part to ω_f of magnitude $0.06|Re(\omega_f)|$ when calculating $w_n(0)$. In reality the damping is likely to be due to g mode losses, that would appear in an improved higher order theory, as well as applied viscosity. With our simple damping prescription, the details of these particular features are not too well reproduced but this is not too surprising in view of the simplifying assumption of an homogeneous core.

7 CONCLUSIONS

We have developed a generalization of the asymptotic treatment of the tidal torques experienced by a massive star as a result of a companion in circular orbit considered by Zahn (1977) and Papaloizou and Savonije (1985) to the case of a rotating star when the forcing frequency is less than the rotation frequency in magnitude. The results confirm the presence of a strong toroidal mode resonance feature for retrograde forcing and also, albeit with a very simplified model of the convective core, the presence of some core inertial mode features found in SP.

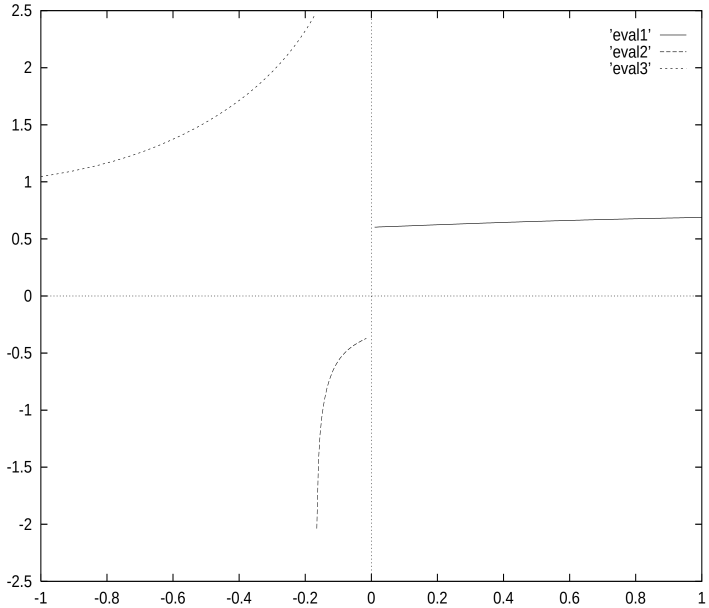

Figure 1. The form of $\log_{10}\lambda_n$ as a function of x^{-1} used in the evaluation of t_0. These λ_n represent the lowest positive values outside the interval (-1/15,0)

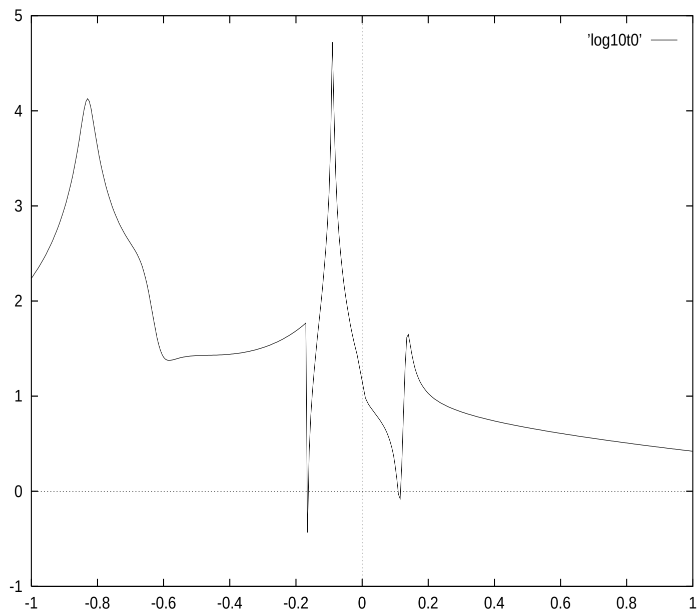

Figure 2. A plot of $\log_{10} t_0$, t_0 in yr., as a function of x^{-1} in the inertial regime for $\Omega_s = 0.1\Omega_c$.

REFERENCES

Chapman S., Lindzen R., 1970, *Atmospheric Tides*, Dordrecht Reidel, Dordrecht, Holland

Goldreich P., Nicholson P.D., 1989. ApJ, 342, 1079

Greenspan H.P., 1968, *The Theory of Rotating Fluids*, Cambridge University press

Lin D.N.C., Papaloizou J.C.B., Kley W.,1993, ApJ, 416, 689

Papaloizou J.C.B., Pringle J.E., 1978, MNRAS, 182, 423

Papaloizou J.C.B., Pringle J.E., 1981, MNRAS, 195, 743

Papaloizou J.C.B., Savonije G.J., 1985, MNRAS, 213, 85

Ryu, D. and Goodman, J. 1992, ApJ, 388, 438

Savonije G.J., Papaloizou J.C.B., 1983, MNRAS, 203, 581

Savonije G.J., Papaloizou J.C.B., 1984, MNRAS, 207, 685

Savonije G.J., Papaloizou J.C.B., Alberts F., 1995, MNRAS, 277, 471

Savonije G.J., Papaloizou J.C.B., 1997, astro-ph/9706186

Tassoul J. L., 1978, *Theory of Rotating Stars*, Princeton University Press, Princeton

Unno W., Osaki Y., Ando H., Saio H., Shibahashi H., 1989, *Non-radial Oscillations of Stars*, University of Tokyo Press, Tokyo

Zahn J.P., 1975, A& A, 41, 329

Zahn J.P., 1977, A& A, 57, 383